Basics of Mathematical Modeling

from the Lecture Notes of Prof. C. Kuttler

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Contents

1 A quick Introduction to Mathematical Modeling

Main idea of mathematical modeling:

1. Abstraction: real world problem (experimental data) is described by a mathematical formulation.
2. Aim: find an appropriate mathematical formulation and use the mathematical tools to investigate the real world phenomenon.
3. No model is THE right one, but only A right one.

A model should be:

- as simple as possible
- as detailed as necessary

There are two ways doing Biomathematics:

Qualitative theory	Quantitative theory
Modeling of the basic mechanisms in a simple way; parameter fitting and analysis of (concrete) data doesn't greatly matter. The results are qualitative. A rigorous analysis of the models is possible and the qualitative results can be compared with experimental results. Quantitative prediction of experimental results is not (main) goal of this approach.	Here, the model of the biological system is very detailed and parameters are taken from the experiments (e.g. by data fitting). The analysis of the system is less important than to get simulations of concrete situations. The results are qualitative, quantitative prediction should be possible. It is important to know a lot of details about the biological system.

Modeling approaches:

Deterministic approach:

1. Difference Equations: The time is discrete, the state (depending on time) can be discrete or continuous. Often used to describe seasonal events.
2. Ordinary Differential Equations (ODEs): Time and state are continuous, space is a homogenous quantities. These approach is often used to describe the evolution of populations.
3. Partial Differential Equations (PDEs): Continuous time and further continuous variables, e.g. space. Used for example to model physical phenomena, like diffusion.

Stochastic approach: Include stochasticity and probability theory in the model. Often used in the context of small populations (also when few data are available).

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2 Linear Models

2.1 Discrete linear models

Time-discrete models means that the development of the system is observed only at discrete times $t_0, t_1, t_2, \ldots$ and not in a continuous time course. Assume here that $t_{k+1} = t_k + h$ where $h > 0$ is a constant step.

An example for discrete linear models is the **Fibonacci equation** (1202), which you probably already know from the highschool. Fibonacci investigated how fast rabbits could breed, assuming that:

- Rabbits are able to mate at the age of one month and at the end of its second month the females can produce another pair of rabbits.
- The rabbits never die.

- The females produce one new pair every month from the second month on.

The Fibonacci sequence is defined by the following recursive formula:

$$x_{n+1} = x_n + x_{n-1}.$$

This equation can be formulated as a 2D discrete-linear system

$$\begin{aligned} x_{n+1} &= x_n + y_n \\ y_{n+1} &= x_n. \end{aligned}$$

Generally, a linear system in 2D can be written as

$$\begin{aligned} x_{n+1} &= a_{11}x_n + a_{12}y_n \\ y_{n+1} &= a_{21}x_n + a_{22}y_n \end{aligned}$$

or in matrix notation

$$\begin{pmatrix} x \\ y \end{pmatrix}_{n+1} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\text{Matrix } A} \begin{pmatrix} x \\ y \end{pmatrix}_n$$

We now look for the **stationary states** of the discrete system. The stationary state for a general discrete system $x_{n+1} = f(x_n)$ is a $\bar{x}$, such that $\bar{x} = f(\bar{x})$. Obviously, as long as we consider linear systems, $(\bar{x}, \bar{y}) = (0, 0)$ is a stationary state.

How to investigate stability.
Consider the system

$$u_{n+1} = Au_n. \tag{1}$$

Then $u_n = A^n u_0$, $n = 0, 1, 2, \ldots$ is the solution of (1) with initial condition u_0. Let λ an eigenvalue of A with the corresponding eigenvector u, then we have $A^n u = \lambda^n u$ and $u_n = \lambda^n u_0$ satisfies the difference equation (1).
For u_0 being a linear combination of eigenvectors of A, $u_0 = b_1 v_1 + \ldots + b_k v_k$, ($\lambda_i$ corresponding eigenvalue of the eigenvector u_i) we get as the solution of (1):

$$u_n = b_1 \lambda_1^n v_1 + \ldots + b_k \lambda_k^n v_k.$$

For the matrix A, the **spectral radius** $\rho(A)$ is defined by

$$\rho(A) := max\{|\lambda| : \lambda \text{ is eigenvalue of } A\}.$$

Theorem 1 *Let A be a $m \times m$ matrix with $\rho(A) < 1$. Then every solution u_n of (1) satisfies*

$$\lim_{n \to \infty} u_n = 0.$$

Moreover, if $\rho(A) < \delta < 1$, then there is a constant $C > 0$ such that

$$\|u_n\| \leq C \|u_0\| \delta^n$$

for all $n \in \mathbb{N}_0$ and any solution of (1).

Remark: If $\rho(A) \geq 1$, then there are solutions u_n of (1) which do not tend to zero for $n \to \infty$. E.g., let λ be an eigenvalue with $|\lambda| \geq 1$ and u the corresponding eigenvector, then $u_n = \lambda^n u$ is a solution of (1) and $\|u_n\| = |\lambda|^n \|u\|$ does not converge to zero for $n \to \infty$.

What happens, if the spectral radius reaches the 1?

Theorem 2 *Let A be a $m \times m$ matrix with $\rho(A) \leq 1$ and assume that each eigenvalue of A with $|\lambda| = 1$ is simple. Then there is a constant $C > 0$ such that*

$$\|u_n\| \leq C\|u_0\|$$

for every $n \in \mathbb{N}$ and $u_0 \in \mathbb{R}^m$, where u_n is solution of (1).

From now on, we consider a **linear two-dimensional discrete system**,

$$u_{t+1} = Au_t, \tag{2}$$

where u_t is a two-dimensional vector and A a real 2×2 matrix (nonsingular).

A "fast formula" for the **computation of the eigenvalues** is

$$\lambda_{1,2} = \frac{1}{2}tr(A) \pm \frac{1}{2}\sqrt{tr(A)^2 - 4det(A)}$$

Let λ be an eigenvalue, then the corresponding eigenvector(s) v, is(are) defined by

$$Av = \lambda v \leftrightarrow (A - \lambda I)v = 0.$$

Theorem 3 *For any real 2×2 matrix A there exists a nonsingular real matrix P such that*

$$A = PJP^{-1},$$

where J is one of the following possibilities

1. $$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

   if A has two real (not necessarily distinct) eigenvalues λ_1, λ_2 with linearly independent eigenvectors.

2. $$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

   if A has a single eigenvalue λ (with a single eigenvector).

3. $$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

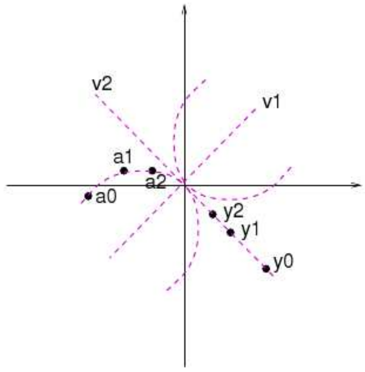
   if A has a pair of complex eigenvalues $\alpha \pm i\beta$ (with non-zero imaginary part)

For real eigenvalues:

Case 1a: $0 < \lambda_1 < \lambda_2 < 1 \Rightarrow$ (0,0) is a **stable node**

All solutions of equation (2) are of the form

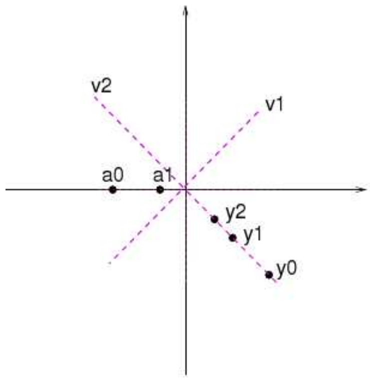
$$u_t = C_1 \lambda_1^t v_1 + C_2 \lambda_2^t v_2,$$



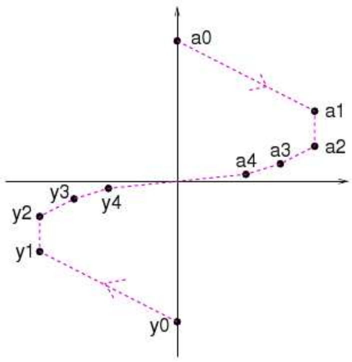
Case 1b: $0 < \lambda_1 = \lambda_2 < 1 \Rightarrow$ (0,0) is a **stable (one-tangent-)node**

There are two possibilities:

If A has one eigenvalue with two independent eigenvectors, case 1a can be slightly modified and the figure has the form

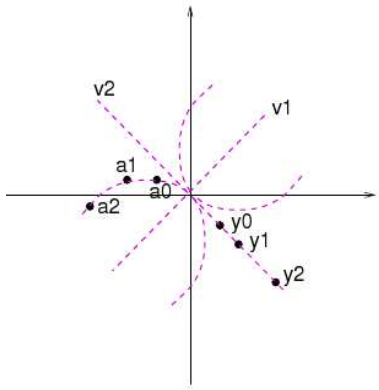


If A has a simple eigenvalue with only one independent eigenvector (and one generalized eigenvector v_2, i.e. $(A - \lambda I)^2 v_2 = 0$), then for $t \to \infty$, all solutions tend to 0.

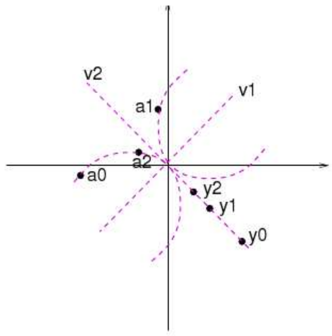


Case 2: $1 < \lambda_1 < \lambda_2 \Rightarrow$ (0,0) is a **unstable node**

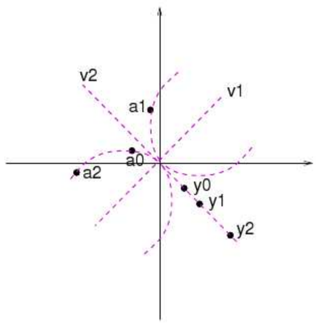
The solutions go away from 0 for $t \to \infty$.



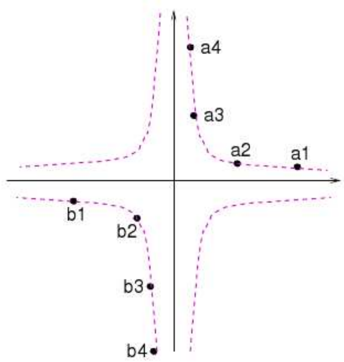
Case 3: $-1 < \lambda_1 < 0 < \lambda_2 < 1 Rightarrow$ (0,0) is a **stable node with reflection**.
Since λ_1^t has alternating signs, the solutions jump between the different branches (provided that $C_1 \neq 0$)



Case 4: $\lambda_1 < -1 < 1 < \lambda_2 \Rightarrow$ (0,0) is an **unstable node with reflection**
Unstable equilibrium, the solutions go away from (0,0), jumping in the direction of v_1.

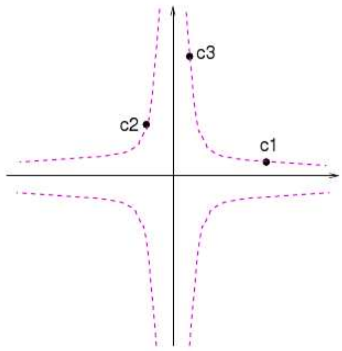


Case 5: $0 < \lambda_1 < 1 < \lambda_2 \Rightarrow$ (0,0) is a **saddle point**



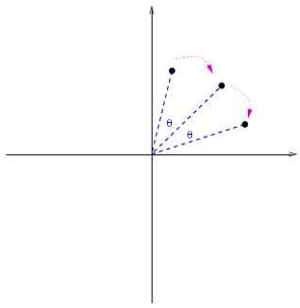
One direction (eigenvector) is stable, the other is unstable.

Case 6: $-1 < \lambda_1 < 0 < 1 < \lambda_2$ (0,0) is a **saddle point with reflection**



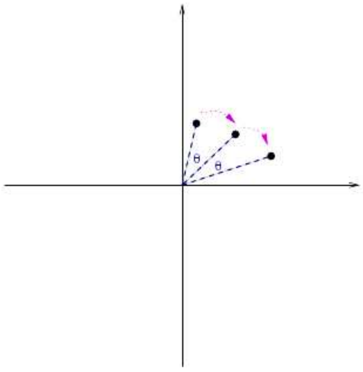
For complex eigenvalues:

Case 7: $\alpha^2 + \beta^2 = 1 \Rightarrow$ (0,0) is a **center**



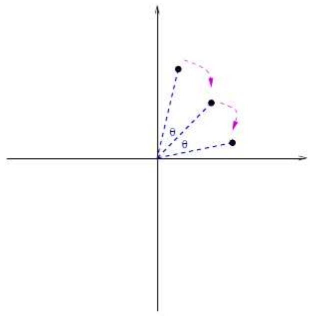
Each solution moves clockwise (with the angle θ) around a circle centred at the origin, which is called a center.

Case 8: $\alpha^2 + \beta^2 > 1 \Rightarrow$(0,0) is an **unstable spiral**



The solution moves away from the origin with each iteration, in clockwise direction.

Case 9: $\alpha^2 + \beta^2 < 1 \Rightarrow$(0,0) is a **stable spiral**



2.2 Continuous linear models

Example for linear ODE model: **Exponential growth**.
The simplest nontrivial IVP for ODEs is

$$\dot{x} = bx$$

with $x(0) = x_0$.
The solution is : $x(t) = x_0 e^{bt}$.
We consider a little generalization:

$$\dot{x} = bx.$$

If $b > 0$, it we have exponential growth, if $b < 0$ exponential decay. b as actual rate can also be interpreted as the difference between an underlying growth rate and an underlying death (or decay) rate.

2.2.1 Existence of solutions of ODEs

Definition 1 (ODE) *An equation $F(t, y(t), y'(t)) = 0$, which relates an unknown function $y = y(t)$ with its derivative $y'(t) = \frac{d}{dt}y(t)$, is called an* ***ordinary differential equation (ODE)*** *of first order, shortly $F(t, y, y') = 0$.*
Often, the explicit case in $\mathbb{R}^n$ is considered:

$$y'(t) = f(t, y), \tag{3}$$

where $f : G \to \mathbb{R}^n$, $G \subseteq \mathbb{R} \times \mathbb{R}^n$ domain.
In the case $n = 1$ it is called scalar.
If f does not depend explicitely on t, the ODE (3) is called autonomous.

Definition 2 *A solution of equation (3) is a function $y : I \to \mathbb{R}^n$, where $I \neq \emptyset$, I interval and*

1. *$y \in C^1(I, \mathbb{R}^n)$ (or y out of another well-suited function space)*
2. *graph$(y) \subset G$*
3. *$y'(t) = f(t, y(t))$ for all $t \in I$.*

Initial value problem: For a given $(t_0, y_0) \in G$ find a solution $y : I \to \mathbb{R}^n$ with

$$\begin{cases} y' = f(t, y) & \text{for all } t \in I \\ y(t_0) = y_0, & t_0 \in I \end{cases} \tag{4}$$

Phase portrait: In the 2D case, $y(t) = (y_1(t), y_2(t))$ is plotted in a (y_1, y_2) coordinate system, the resulting curves are parametrised by t.

Theorem 4 (Peano, 1890) ***(Existence theorem-only!)*** *Let $f \in C(G, \mathbb{R}^n)$ and $(t_0, y_0) \in G$. Then, there exists an $\varepsilon > 0$ and a solution $y : I \to \mathbb{R}^n$ of the initial value problem (4) with $I = [t_0 - \varepsilon, t_0 + \varepsilon]$ and graph$(y) \subset G$.*

Theorem 5 (Picard-Lindelöf) *Let $f \in C^{0,1-}(G, \mathbb{R}^n)$ and $(t_0, y_0) \in G$. Then, there exists an $\varepsilon > 0$ and a solution $y : I \to \mathbb{R}^n$ of the initial value problem (4) with $I = [t_0 - \varepsilon, t_0 + \varepsilon]$ and graph$(y) \subset G$. Furthermore, y is determined uniquely in I, i.e. for any solution $z : J \to \mathbb{R}^n$ of the initial value problem is $y|_{I\cap J} = z|_{I\cap J}$.*

Remark: f is called **Lipschitz continuous**(indicated by C^{1-}) in G with respect to y, if there exists a constant $L > 0$ with

$$\|f(t, y) - f(t, z)\| \leq L\|y - z\| \quad \text{for all } (t, y), (t, z) \in G.$$

Definition 3 *A solution $y : I \to \mathbb{R}^n$ of the initial value problem (4) is called maximal, if there is no solution $z : J \to \mathbb{R}^n$ with $I \subset J$ and $z|_I = y$. The interval I is open: $I = (a, b)$.*

Existence of a maximal solution of the initial value problem (4) can be shown by the lemma of Zorn.

Proposition 1 *Let $G = \mathbb{R} \times \mathbb{R}^n$, $(t_0, y_0) \in G$. Let $y : (a, b) \to \mathbb{R}^n$ be the maximal solution of the initial value problem (4). If $b = \infty$, then the solution exists for all $t > t_0$ (global existence). If $b < \infty$, then we have*

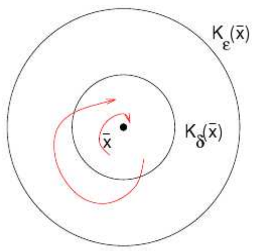
$$\|y(t)\| \to \infty \quad \textit{for } t \to b_-.$$

2.2.2 Stability and attractiveness

Let $f \in C^{0,1-}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $\dot{x} = f(t, x)$, let $x(t, t_0, x_0)$ be the maximal solution with $x(t_0) = x_0$. Let $\bar{x} \in C^1([t_0, \infty), \mathbb{R}^n)$ be a fixed solution of the ODE with $\bar{x}(t_0) = \bar{x}_0$. All solutions exist globally in a neighborhood of $\bar{x}$, for given initial data.

Definition 4 (Stability, Lyapunov) *$\bar{x}$ is called stable, if*

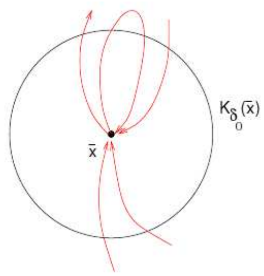
$$\forall\, t_1 \geq t_0 \;\; \forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 : \;\; (\|x_0 - \bar{x}_0\| < \delta \;\; \Rightarrow \;\; \|x(t, t_1, x_0) - \bar{x}(t)\| < \varepsilon \textit{ in } [t_1, \infty))$$



$\bar{x}$ is called uniformly stable, if δ does not depend on t_1 (which is always the case if f is not dependent on or periodic in t).

Definition 5 (Attractiveness) *$\bar{x}$ is called attractive, if*

$$\forall\, t_1 \geq t_0 \;\; \exists\, \delta_0 > 0 : \;\; (\|x_0 - \bar{x}_0\| < \delta_0 \;\; \Rightarrow \;\; \lim_{t\to\infty} \|x(t, t_1, x_0) - \bar{x}(t)\| = 0).$$



In this case, all solutions starting in the δ_0 neighborhood of $\bar{x}(t_1)$ are "attracted" by $\bar{x}$. $\bar{x}$ is called attractor for that trajectory.
Domain of attraction:

$$\mathcal{A}(\bar{x}) = \{x_0 \in \mathbb{R}^n \,|\, \exists\, t_1 \geq t_0 : \;\; \lim_{t\to\infty} \|x(t, t_1, x_0) - \bar{x}(t)\| = 0\}$$

$\bar{x}$ is called uniformly attractive, if δ_0 does not depend on t_1 (which is the case if f is not dependent on or periodic in t).
$\bar{x}$ is called asymptotically stable, if $\bar{x}$ is stable and attractive.

2.2.3 Continuous linear systems with constant coefficients

Let $A \in \mathbb{C}^{n\times n}$, and consider the linear system of ODEs

$$\dot{x} = Ax \text{ in } I = \mathbb{R}. \tag{5}$$

We have:

$$x(t) = e^{\lambda t}a \text{ is solution of (5)} \;\; \Leftrightarrow \;\; a \text{ is eigenvector of } A \text{ corr. to the eigenvalue } \lambda.$$

If A is real and $\lambda = \mu + i\nu$ is an eigenvalue of A with eigenvector $a + ib$, then the complex solution $x(t) = e^{\lambda t}(a + ib)$ yields two real solutions by $z_1 = Re\, x$ and $z_2 = Im\, x$.
The stationary points of the system $\dot{x} = Ax$ are given by the vectors in the kernel of A.
In particular, if $det\, A \neq 0$, then there is only the stationary point 0.

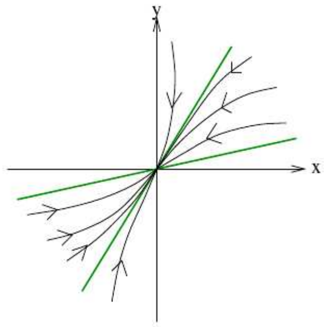
2.2.4 Special case: Linear 2×2 systems

Let $A \in \mathbb{R}^{2\times 2}$ and $\dot{z}(t) = Az(t)$, where $z(t) = (x(t), y(t))^T$. Let $(x(t), y(t))$ be a solution curve of the autonomous 2D system. Each point (x, y) is assigned to a vector in the tangent field.

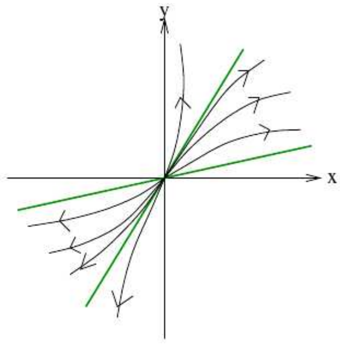
The qualitative dynamic behavior (of the solution curves) depends on the eigenvalues.
Let $\lambda_1, \lambda_2 \neq 0$, $\lambda_1 \neq \lambda_2$.

Case 1 λ_1, λ_2 are real, $\lambda_1 \cdot \lambda_2 > 0$ (i.e. they have the same sign)

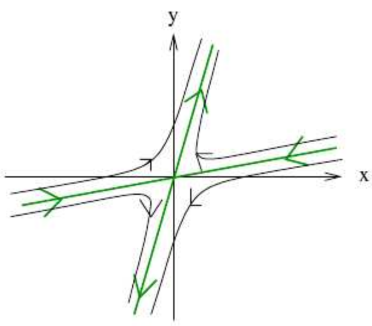
a) $\lambda_1, \lambda_2 < 0$. Then $det(A) > 0$, $tr(A) < 0$
$\Rightarrow (0,0)$ is a stable, two-tangents node.



b) $\lambda_1, \lambda_2 > 0$. Then $det(A) > 0$, $tr(A) > 0$ (compare to a) with inverse time)
$\Rightarrow (0,0)$ is an unstable, two-tangents node



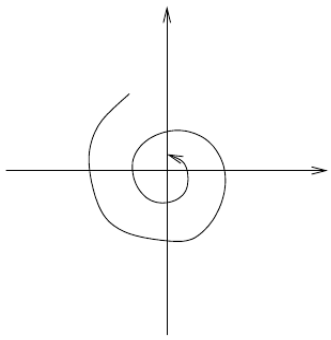
Case 2 $\lambda_1 < 0$, $\lambda_2 > 0$. Then $det(A) < 0$.
This means: There is one stable and one unstable “direction”.



$\Rightarrow (0,0)$ is a saddle.

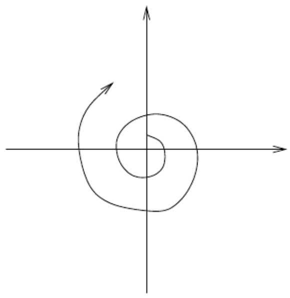
Case 3 $\lambda_1 = \bar{\lambda}_2$ with non-vanishing imaginary part, the eigenvalues are complex conjugated. Then $tr(A)^2 < 4det(A), det(A) > 0$

a) $Re(\lambda_1) = Re(\lambda_2) < 0$ ($\rightsquigarrow tr(A) < 0$)



$\Rightarrow (0,0)$ is a stable spiral.

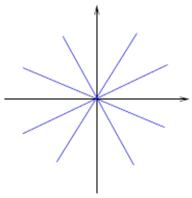
b) $Re(\lambda_1) = Re(\lambda_2) > 0$, then $tr(A) > 0$. This case is analogous to a) with inverse time.



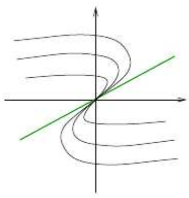
$\Rightarrow (0,0)$ is an unstable spiral.

Further cases / special cases

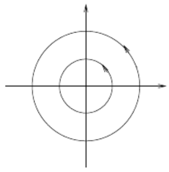
- both eigenvalues are real and equal:
  - there are two linearly independent eigenvectors
    $\Rightarrow (0,0)$ is a star (positive eigenvalues: unstable; negative eigenvalues: stable)



  - there is only one eigenvector (stability depending on the sign)
    $\Rightarrow (0,0)$ is a one-tangent node.

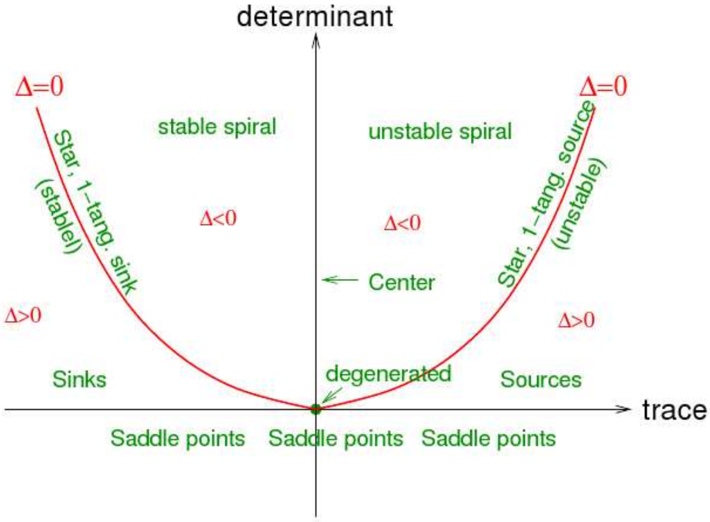


- the eigenvalues are purely imaginary
  $\Rightarrow (0,0)$ is a centre



At least one eigenvalue is $= 0$: "degenerated"

The results are summarized in the so called **trace-determinant diagram** for linear 2×2 systems with constant coefficients:



2.2.5 General linear systems

For the stability of a linear system higher dimension:

Proposition 2 *Consider the linear case $\dot{x} = Ax$, $A \in \mathbb{C}^{n \times n}$. Let $\sigma(A)$ be the spectrum of A.*

1. *0 is asymptotically stable $\Leftrightarrow$ Re $\sigma(A) < 0$*
2. *0 is stable $\Leftrightarrow$ Re $\sigma(A) \leq 0$ and all eigenvalues λ with Re $\lambda = 0$ are semi-simple (i.e., geometric and algebraic multiplicity are the same)*
3. *If there is a $\lambda \in \sigma(A)$ with Re $\lambda > 0$, then 0 is unstable. (The reversed direction is wrong!!!)*

3 Nonlinear models

3.1 Discrete nonlinear models

The general nonlinear one-dimensional difference equation of first order is:

$$x_{n+1} = f(x_n). \tag{6}$$

The discrete linear equation (1) can be considered as a special case of (6).

We introduce now some general concepts:

Definition 6 *$\bar{x}$ is called **stationary point** of the system $x_{n+1} = f(x_n)$, if*

$$\bar{x} = f(\bar{x}).$$

$\bar{x}$ is also called fixed point or steady state.

An example:

$$x_{n+1} = ax_n + b \qquad \text{(i.e. } f(x_n) = ax_n + b), \tag{7}$$

where
a: constant reproduction rate; growth / decrease is proportional to x_n (Assumption: $a \neq 1$)
b: constant supply / removal.

Definition 7 *An **autonomous discrete nonlinear system** is given by*

$$u_{n+1} = f(u_n), \qquad n \in \mathbb{N}_0, \tag{8}$$

where $u_n \in \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m$ (or $f : D \to D,\ D \subseteq \mathbb{R}^m$).

If A is a $m \times m$ matrix, then the linear system $f(x) = Ax$ is a special case of (8).

3.1.1 Analysis of discrete nonlinear dynamical systems

One of our main tasks is the investigation of the behavior of x_n "in the long time run", i.e., for large n.
We look for stationary points of equation (7):

$$\begin{aligned} f(\bar{x}) = \bar{x} \quad &\Leftrightarrow \quad a\bar{x} + b = \bar{x} \\ &\Leftrightarrow \quad b = (1-a)\bar{x} \\ &\Leftrightarrow \quad \bar{x} = \frac{b}{1-a} \end{aligned}$$

Hence, there exists exactly one stationary state of (7).

Definition 8 *Let $\bar{x}$ be a stationary point of the system $x_{n+1} = f(x_n)$.*
*$\bar{x}$ is called **locally asymptotically stable** if there exists a neighborhood U of $\bar{x}$ such that for each starting value $x_0 \in U$ we get:*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

*$\bar{x}$ is called **unstable**, if $\bar{x}$ is not (locally asymptotically) stable.*

How to investigate stability of stationary points?
Consider a stationary point $\bar{x}$ of the difference equation $x_{n+1} = f(x_n)$. We are interested in the local behavior near $\bar{x}$. For this purpose, we consider the deviation of the elements of the sequence to the stationary point $\bar{x}$:

$$z_n := x_n - \bar{x}$$

z_n has the following property:

$$\begin{aligned} z_{n+1} &= x_{n+1} - \bar{x} \\ &= f(x_n) - \bar{x} \\ &= f(\bar{x} + z_n) - \bar{x}. \end{aligned}$$

Let the function f be differentiable in $\bar{x}$, thus we get $\lim_{h\to 0} \frac{f(\bar{x}+h)-f(\bar{x})}{h} = f'(\bar{x})$ and $f(\bar{x}+h) = f(\bar{x}) + h \cdot f'(\bar{x}) + O(h^2)$. This yields:

$$\begin{aligned} z_{n+1} &= f(\bar{x} + z_n) - \bar{x} \\ &= f(\bar{x} + z_n) - f(\bar{x}) \\ &= z_n \cdot f'(\bar{x}) + O(z_n^2). \end{aligned}$$

$O(z_n^2)$ is very small and can be neglected, i.e. we approximate the nonlinear system $x_{n+1} = f(x_n)$ by

$$z_{n+1} \approx z_n \cdot f'(\bar{x}),$$

which is again a linear difference equation, for which we already know the stability criteria.

Proposition 3 *Let f be differentiable. A stationary point $\bar{x}$ of $x_{n+1} = f(x_n)$ is*

- *locally asymptotically stable, if $|f'(\bar{x})| < 1$*
- *unstable, if $|f'(\bar{x})| > 1$*

Remark: These criteria are sufficient, but not necessary!

In case of the non-homogeneous, linear system we have $f'(\bar{x}) = a$, which means that the stationary point is locally asymptotically stable if $|a| < 1$ (respectively, unstable, if $|a| > 1$).

Definition 9 *Let $u_{n+1} = f(u_n)$ be an autonomous system, $f : D \to D$, $D \subseteq \mathbb{R}^m$. A vector $v \in D$ is called **equilibrium or steady state** or stationary point or fixed point of f, if $f(v) = v$ and $v \in D$ is called periodic point of f, if $f^p(v) = v$. p is a period of v.*

1. *Let $v \in D$ be a fixed point of f. Then v is called stable, if for each $\varepsilon > 0$ there is $\delta > 0$ such that*

   $$\|f^n(u) - v\| < \varepsilon \quad \text{for all } u \in D \text{ with } \|u - v\| < \delta \text{ and all } n \in \mathbb{N}_0$$

   (i.e. $f^n(U_\delta(v)) \subseteq U_\varepsilon(v)$). If v is not stable, it is called unstable.

2. *If there is, additionally to 1., a neighborhood $U_r(v)$ such that $f^n(u) \to v$ as $n \to \infty$ for all $u \in U_r(v)$, then v is called asymptotically stable.*

3. *Let $w \in D$ be a periodic point of f with period $p \in \mathbb{N}$. Then w is called (asymptotically) stable, if w, $f(w), \ldots, f^{p-1}(w)$ are (asymptotically) stable fixed points of f^p.*

Remark: Intuitively, a fixed point v is stable, if points close to v do not move far from v. If additionally all solutions starting near v converge to v, v is asymptotically stable.

Theorem 6 *Let $u_{n+1} = f(u_n)$ be an autonomous system. Suppose $f : D \to D$, $D \subseteq \mathbb{R}^m$ open, is twice continuously differentiable in some neighborhood of a fixed point $v \in D$. Let J be the Jacobian matrix of f, evaluated at v. Then*

1. *v is asymptotically stable if all eigenvalues of J have magnitude less than 1.*
2. *v is unstable if at least one eigenvalue of J has magnitude greater than 1.*

Remark: If $max\{|\lambda| : \lambda \text{ eigenvalue of } K\} = 1$, then we cannot give a statement about the stability of the fixed point v by that criterion; the behavior then depends on higher order terms than linear ones.

3.1.2 The 2D case

Here we consider the 2D case more concrete. The discrete system can be formulated with the variables x and y:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned} \tag{9}$$

Stationary states $\bar{x}$ and $\bar{y}$ satisfy

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{y}) \\ \bar{y} &= g(\bar{x}, \bar{y}) \end{aligned}$$

We need the Jacobian matrix at a certain stationary point $(\bar{x}, \bar{y})$:

$$A = \begin{pmatrix} \frac{\partial f}{\partial x}|_{\bar{x},\bar{y}} & \frac{\partial f}{\partial y}|_{\bar{x},\bar{y}} \\ \frac{\partial g}{\partial x}|_{\bar{x},\bar{y}} & \frac{\partial g}{\partial y}|_{\bar{x},\bar{y}} \end{pmatrix}$$

The eigenvalues λ_1 and λ_2 of A yield the information about stability of the system. In some cases, it is easier to handle the following (necessary and sufficient) condition:
Both eigenvalues satisfy $|\lambda_i| < 1$ and the steady state $(\bar{x}, \bar{y})$ is stable, if

$$2 > 1 + det\, A > |tr\, A|. \tag{10}$$

This can be easily shown:
The characteristic equation reads

$$\lambda^2 - tr\, A\, \lambda + det\, A = 0$$

and has the roots

$$\lambda_{1,2} = \frac{tr\, A \pm \sqrt{tr^2 A - 4\, det\, A}}{2}$$

In case of real roots, they are equidistant from the value $\frac{tr\, A}{2}$. Thus, first has to be checked that this midpoint lies inside the interval $(-1, 1)$:

$$-1 < \frac{tr\, A}{2} < 1 \quad \Leftrightarrow \quad |tr\, A/2| < 1.$$

Furthermore, the distance from $tr\, A/2$ to either root has to be smaller than to an endpoint of the interval, i.e.

$$1 - |tr\, A/2| > \frac{\sqrt{tr^2 A - 4\, det\, A}}{2}.$$

Squaring leads to

$$1 - |tr\,A| + \frac{tr^2 A}{4} > \frac{tr^2 A}{4} - det\,A,$$

and this yields directly

$$1 + det\,A > |tr\,A|.$$

Advantage: It is not necessary to compute explicitely the eigenvalues

3.2 Continuous nonlinear models

An example of continuous nonlinear dynamical system is the classical predator prey model, introduced by Volterra and Lotka (1925/26)

$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= -dy + cxy, \end{aligned} \tag{11}$$

where $x(t)$ are the prey and $y(t)$ are the predators at time t and all parameters are real and larger that zero.
We compute the stationary points of (11):

$$\dot{x} = ax - bxy = 0 \quad \Leftrightarrow \quad x = 0 \text{ or } y = \frac{a}{b}$$
$$\dot{y} = cxy - dy = 0 \quad \Leftrightarrow \quad y = 0 \text{ or } x = \frac{d}{c}$$

$\Rightarrow$ $(0, 0)$ and $(\frac{d}{c}, \frac{a}{b})$ are stationary points. $(\frac{d}{c}, \frac{a}{b})$ is called "coexistence point". Analyse the stability by a linearization.
Let $(\bar{x}, \bar{y})$ be a stationary point. A perturbation is considered:

$$x = \bar{x} + u, \quad y = \bar{y} + v,$$

this yields

$$\begin{aligned} \frac{d}{dt}x = \frac{d}{dt}(\bar{x} + u) &= f(\bar{x} + u, \bar{y} + v) = \underbrace{f(\bar{x}, \bar{y})}_{=0} + \frac{\partial f(\bar{x}, \bar{y})}{\partial x}u + \frac{\partial f(\bar{x}, \bar{y})}{\partial y}v + \dots \\ \frac{d}{dt}y = \frac{d}{dt}(\bar{y} + v) &= g(\bar{x} + u, \bar{y} + v) = \underbrace{g(\bar{x}, \bar{y})}_{=0} + \frac{\partial g(\bar{x}, \bar{y})}{\partial x}u + \frac{\partial g(\bar{x}, \bar{y})}{\partial y}v + \dots \end{aligned}$$

$\rightsquigarrow$ Approximation with linear terms:

$$\begin{aligned} \dot{u} &= \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v \\ \dot{v} &= \frac{\partial g}{\partial x}u + \frac{\partial g}{\partial y}v \end{aligned}$$

This can also be done more generally, for dimension n. In the frame of

$$\dot{x} = f(x), \quad f \in C^1(\mathbb{R}^n, \mathbb{R}^n), \quad f(\bar{x}) = 0, \;\; \bar{x} \in \mathbb{R}^n, \tag{12}$$

we consider solutions $x(t)$ of (12) in the neighborhood of $\bar{x}$, $x(t) = \bar{x} + y(t)$, then

$$\dot{y}(t) = f'(\bar{x})y(t) + o(\|y\|).$$

The corresponding linearized system is

$$\dot{z} = Az, \quad A = f'(\bar{x}) = \left(\frac{\partial f_i}{\partial x_k}(\bar{x})\right).$$

Proposition 4 (Linearization, Stability, Perron, Lyapunov) *If the real parts of all eigenvalues of $A = f'(\bar{x})$ are negative, then $\bar{x}$ is exponentially asymptotically stable, i.e. there are constants $\delta, C, \alpha > 0$, such that $\|x(0) - \bar{x}\| < \delta$ implies*

$$\|x(t) - \bar{x}\| < Ce^{-\alpha t} \qquad \textit{for } t \geq 0.$$

Addendum:
From $Re\, \sigma(A) \cap (0, \infty) \neq \emptyset$ it follows that $\bar{x}$ is unstable.

When do linear and nonlinear model "correspond" locally?

Definition 10 *$\bar{x}$ is called hyperbolic, if $0 \notin Re\, \sigma(f'(\bar{x}))$.*

Proposition 5 (Hartman & Grobman, 1964) *Let $\bar{x}$ be hyperbolic. Then, there is a neighborhood U of $\bar{x}$ and a homeomorphism $H : U \to \mathbb{R}^n$ with $H(\bar{x}) = 0$, which maps the trajectories of $\dot{x} = f(x)$ one-to-one into trajectories of $\dot{z} = Az$, with respect to the time course.*

3.2.1 Proceeding in the 2D case

Let $Re\, \lambda_j \neq 0$ for all j. Then all solution curves of the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

show the same qualitative behavior at the stationary point $(\bar{x}, \bar{y})$ as those of the corresponding linear problem.

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

(derivatives at $(\bar{x}, \bar{y})$)

Remark: This is not valid for $Re\, \lambda = 0$, i.e. there are problems with the examination of centres and spirals.

Example: application of the theory to the predator-prey model (11).
General Jacobian Matrix:

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} a - by & -bx \\ cy & cx - d \end{pmatrix}$$

- In $(0,0)$, the Jacobian is $\begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}$
  $\Rightarrow$ eigenvalues a, $-d$
  $\Rightarrow$ saddle point (unstable)

- In $(\frac{d}{c}, \frac{a}{b})$, the Jacobian is: $\begin{pmatrix} a - b\frac{a}{b} & -b\frac{d}{c} \\ c\frac{a}{b} & c\frac{d}{c} - d \end{pmatrix} = \begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ca}{b} & 0 \end{pmatrix}$
  $tr = 0$, $det = \frac{abcd}{bc} = ad$, thus:

$$\lambda_{1,2} = \frac{tr}{2} \pm \sqrt{\frac{tr^2}{4} - det} = \pm i\sqrt{ad} \qquad \text{(purely imaginary)}$$

  $\Rightarrow$ no statement about stability possible at the moment
  We do not know now, if $(\frac{d}{c}, \frac{a}{b})$ is a spiral or if there are closed trajectories (solution curves)!

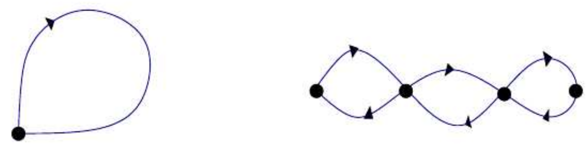
3.2.2 Further results for 2D ODE systems

For (linear and) nonlinear ODE two-dimensional systems, there are two important results.

Theorem 7 (Poincare-Bendixson) *Consider a trajectory $x(t) \in \mathbb{R}^2$ (or $x(t) \in D$, where $D \subset \mathbb{R}^2$ is compact and connected, positively invariant) of the ODE $\dot{x} = f(x)$, f smooth, with only finitely many roots. If $x(t)$ is bounded, then the ω-limit set is one of the following objects:*

- *a stationary point*
- *a periodic orbit*
- *a homoclinic orbit or heteroclinic cycle.*

A **homoclinic orbit** is an orbit that tends to the same stationary point for $t \to \pm\infty$, while a heteroclinic orbit tends to different stationary points. A **heteroclinic cycle** is a closed chain of heteroclinic orbits. Somehow, a homoclinic orbit resp. a heteroclinic cycle can be interpreted as a generalization of a periodic orbit.



homoclinic orbit heteroclinic cycle

Direct consequence of this theorem: If there is no stationary point, there has to be a periodic orbit.

Remark: This proposition is not valid e.g. in higher dimensional spaces (famous counterexample: The Lorenz attractor)

In some cases, the existence of closed orbits can be excluded by the so-called negative criterion of Bendixson-Dulac:

Proposition 6 (Negative criterion of Bendixson-Dulac) *Let* $D \subseteq \mathbb{R}^2$ *be a simply connected region and* $(f, g) \in C^1(D, \mathbb{R})$ *with div* $(f, g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ *being not identically zero and without change of sign in* D*. Then the system*

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

has no closed orbits lying entirely in D*.*

References

[1] Amann, H. *Gewöhnliche Differentialgleichungen.* de Gruyter (1983).

[2] Braun, M. *Differential Equations and their applications.* Springer (1975).

[3] Guckenheimer, J. and Holmes, P., *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Copernicus (1990).

[4] Hale, J., *Ordinary differential equations*, Dover (2009).

[5] Holmgren, R. *A First Course in Discrete Dynamical Systems.* Springer-Verlag, New York (1994).